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Algebra 1

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Quiz #1

Problems:

1. Use mathematical induction to prove that $7n + 5 < 2^n$ for every integer $n \geq 6$.
   Solutions: Let $P(n)$ denote this inequality (where $n \in N$). We use induction to prove: $P(n)$ for every integer $n \geq 6$.
   Base step: For $n = 6$, we have
   $$7n + 5 = 7 \cdot 6 + 5 = 47 < 64 = 26 = 2n$$
   Hence, $P(6)$ is true.
   Inductive step: Now suppose that that $P(k)$ holds for some integer $k \geq 6$. Then $7k + 5 < 2^k$. We must now verify that $P(k+1)$ is true. Note that since $k \geq 6$, we have $7 < 64 \leq 2^k$ (so $7 < 2^k$). Then by $P(k)$, we have
   $$7(k+1) + 5 = 7k + 5 + 7 < 2^k + 7 < 2^k + 2^k = 2^{k+1}$$
   Hence, $P(k+1)$ is true (note that we used $7 < 2^k$, for the second inequality). Therefore, by the principle of mathematical induction, $7n + 5 < 2^n$ for every integer $n \geq 6$.

2. Let R be the relation defined on $\mathbb{Z}$ by $xRy\ if\ |x - y| \leq 3$. Which of the properties, reflexive, symmetric and transitive, does the relation R possess? Is R an equivalence relation? Justify your answers.
   Solutions: We will show that R is reflexive and symmetric, but not transitive. Let $x \in \mathbb{Z}$, then
   $$|x - x| = 0 \leq 3,$$
   so xRx; hence, R is reflexive. Let $x, y \in \mathbb{Z}$ and suppose that xRy. Then $|x-y| \leq 3$. We have
   $$|y - x| = |x - y| \leq 3,$$
   so yRx; hence, R is symmetric. We show that R is not transitive by counterexample. Let $x = 0$, $y = 2$ and $z = 4$. Then
   $$|x - y| = |0 - 2| = 2 \leq 3$$
   and
   $$|y - z| = |2 - 4| = 2 \leq 3,$$
   so xRy and yRz. But observe that
   $$|x - z| = |0 - 4| = 4 > 3$$
   so x and z are not in relation. So R is not transitive. Therefore, R is not an equivalence relation.

3. (a) Find u, $v \in \mathbb{Z}$, such that $gcd(50,7) = 50u + 7v$.
   Solutions:
   $$50 - 7 \cdot 7 = 1$$
   So $u = 1$ and $v = -7$.

   (b) Is $[7]_{50}$ in the group of units mod 50, U_{50}? If yes give it's multiplicative inverse.
   Solutions: From the previous question.
   $$[7]_{50}[-7]_{50} = [1]$$
   So $[7]_{50} \in U_{50}$ and the multiplicative inverse of $[7]_{50}$ is $[-7]_{50}$.

   (c) Is $[5]_{50}$ in the group of units *mod* 50, U_{50}?. Justify your answer.
   Solutions: $[5]_{50} \cdot [10]_{50} = [0]$ so $[5]_{50}$ is a zero divisor in $\mathbb{Z}/50\mathbb{Z}$, it cannot be invertible.

4. Is the equality $[a]_3^{[b]_3} := [a^b]_3$ well defined in $\mathbb{Z}/3\mathbb{Z}$.
   Solutions: No, since for $a = 2$ and $b = 2$ then $[2^2]_3 = [1]_3$ and $[b]_3 = [5]_3$ and $[2^5]_3 = [2]_3$ and $[2^5] \neq [2^2]$.